New Questions for Junior High Number Sense (2003)

This document contains information on some of the new tricks that will be appearing on the 2003 District and State Number Sense tests. Some of these tricks are not new and students familiar with the Texas Math and Science Coaches Association (TMSCA) contests will recognize many of the following topics.

The new tricks are sectioned in the order that they will appear on the test. Some of the new tricks are given implicitly. For the others, the student is encouraged to search for an easy mental math formula or procedure for working the problem.

The product of four consecutive integers, plus one. [#20-40] For example, $3 \cdot 4 \cdot 5 \cdot 6 + 1$. To find the solution, one could multiply all four numbers together and add one, however, there is a short-cut. The solution can be found by multiplying the first and last number in the sequence, adding one, and squaring the result. Thus, my example yields $(3 \cdot 6 + 1)^2 = (18 + 1)^2 = 19^2 = 361$. (Verify!)

An advanced problem [#60-80] could be asked by giving the result and asking for the smallest (or largest) of the four consecutive integers. For example, "The product of four consecutive positive integers is 360. Find the smallest of these integers." Since the product of four consecutive integers plus one is always a perfect square, add one to 360 to get 361. Since $361 = 19^2$, we know that the product of the smallest and largest integers, plus one is 19. Algebraically, that gives $x(x + 3) + 1 = 19$. This equation simplifies to $x^2 + 3x - 18 = 0$. Factoring gives $(x + 6)(x - 3) = 0$ and the roots are -6 and 3. Since the question stated that only positive numbers are used, we exclude -6 and the answer is 3.

Multiplication by 167. [#40-60] This one is obvious. I'll let you think about the trick to this problem. As a starting point, know that the questions will only ask you to multiply multiples of 6 by 167. Good luck.

Factorials, Permutations, and Combinations. [#60-80] Questions of these types will consist of simply computing factorials, permutations, and combinations. Examples include $5!$, $P(6, 2)$, and $C(7, 3)$.

Factorials: As a basis of learning, everyone should know the factorials from 0 to 10. The standard symbol (!) is used to represent factorials. The factorial of a positive integer N is defined as the product of every positive integer from N down to 1. Therefore,

$$N! = N \cdot (N - 1) \cdot (N - 2) \cdots 3 \cdot 2 \cdot 1.$$

With this, we have $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Permutations: The number of permutations of a set (or group of items) is the number of ways in which the items can be re-ordered when the order of the items is meaningful. If the original set has n elements, the permutation $P(n, k)$ gives the number of ways k elements from the original n elements of the set can be re-arranged into a particular order. This is given by the formula

$$P(n, k) = \frac{n!}{(n - k)!}.$$

Therefore,

$$\begin{aligned} P(6,2) &= \frac{6!}{(6-2)!} \\ &= \frac{6!}{4!} \\ &= \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} \\ &= 6\cdot5 \\ &= 30. \end{aligned}$$

Notice that the short-cut to computing the permutation $P(n,k)$ is to multiply the sequence of k integers, starting with n and working your way down towards 1. From above, note that $P(6,2) = 6\cdot5$. That is the product of 2 integers, starting with 6 and working your way down towards 1. With only 2 integers to consider, you don't get far! Another example,

$$\begin{aligned} P(9,4) &= \underbrace{9\cdot8\cdot7\cdot6}_{\text{4 numbers}} \\ &= 3024 \end{aligned}$$

Combinations: Combinations are similar to permutations, except combinations are *unordered*. Thus, for combinations, the order of the re-arrangements does not matter. Combinations are given by the formula

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

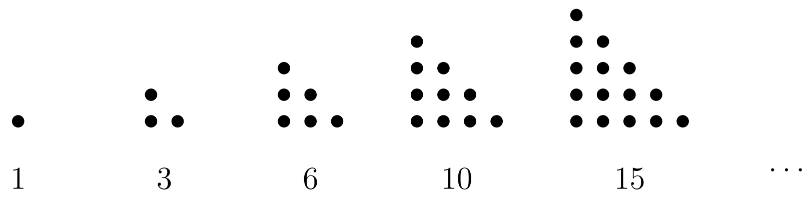
Notice that this is the same as the permutation $P(n,k)$ divided by $k!$. This division by $k!$ removes the multiple counting of sets that have the same elements, but are in a different order. To calculate $C(7,3)$, begin with the permutation $P(7,3)$ and divide by $3!$.

$$\begin{aligned} C(7,3) &= \frac{P(7,3)}{3!} \\ &= \frac{7\cdot6\cdot5}{3\cdot2\cdot1} \\ &= 7\cdot5 \\ &= 35 \end{aligned}$$

More Square Roots [#60-80] Some questions on the number sense tests will focus on working with square roots, including reducing square roots completely and multiplying terms with square roots. For example, simplify $\sqrt{48}$. A square root is NOT simplified completely if there is a perfect square (greater than 1) that is a factor of the number inside the square root. This square root ($\sqrt{48}$) is not simplified since $48 = 16 \cdot 3$ and $16 = 4^2$. Simplifying, we get $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$.

Also, to simplify $(3 - \sqrt{2})(3 + \sqrt{2})$, we must use FOIL. $(3 - \sqrt{2})(3 + \sqrt{2}) = 9 + 3\sqrt{2} - 3\sqrt{2} - \sqrt{4}$. Notice that the middle terms cancel each other out. This happens because the only difference in the two terms is the middle sign. Also, change $\sqrt{4}$ to 2. This leaves us with $9 - 2 = 7$. The answer is 7.

Triangular Numbers [#60-80] The triangular numbers are the numbers that result from building successively larger triangles, as shown in the pattern below.



Let T_n represent the nth triangular number, meaning that T_1 is the first triangular number, T_2 is the second triangular number, and so on. Then, the formula for finding the nth triangular number is $T_n = \frac{n(n+1)}{2}$. For example, "what is the 6th triangular number?" To find the answer, substitute $n = 6$ into the formula. Thus, we have $T_6 = \frac{6(6+1)}{2} = 3(7) = 21$.

To find the sum of two consecutive triangular numbers, $T_{n-1} + T_n$, the result is simply n^2. That is, you take the higher index of the two triangular numbers and square it! For example, "What is the sum of the sixth and seventh triangular numbers?" The answer is $7^2 = 49$ because the indexes of these two triangular numbers are 6 and 7. Since 7 is higher, take 7 and square it.

To find the (positive) difference of two consecutive triangular numbers, $T_n - T_{n-1}$, the answer is simply n. So, "What is the (positive) difference between the eighth and ninth triangular number?" The answer must be 9, since 9 is the higher index.

Finally, the product of two consecutive triangular numbers, $T_{n-1} \cdot T_n$, is found by the formula

$$T_{n-1} \cdot T_n = \frac{1}{4}n^2(n^2 - 1).$$

So, if the question asks, "What is the product of the ninth and tenth triangular numbers?", use the larger index, 10, and the formula:

$$
\begin{aligned}
T_9 \cdot T_{10} &= \frac{1}{4}(10^2)(10^2 - 1) \\
&= \frac{1}{4}(100)(99) \\
&= (25)(99) \\
&= 2475
\end{aligned}
$$

PRACTICE QUESTIONS – The following practice questions cover the above examples and should be used to guide your inquiries into the new types of questions to be asked on the number sense tests.

1. $2 \cdot 3 \cdot 4 \cdot 5 + 1 =$
2. $5 \cdot 6 \cdot 7 \cdot 8 + 1 =$
3. $4! =$
4. $7! =$
5. $P(8, 2) =$
6. $P(5, 3) =$
7. $C(8, 3) =$
8. $C(5, 2) =$
9. Simplify $\sqrt{50}$.
10. Simplify $\sqrt{72}$.
11. $(5 - \sqrt{3})(5 + \sqrt{3}) =$
12. $(8 + \sqrt{10})(8 - \sqrt{10}) =$
13. What is the ninth triangular number?
14. What is the fifth triangular number?
15. What is the sum of the seventh and eighth triangular numbers?
16. What is the sum of the 12th and 13th triangular numbers?
17. What is the positive difference between the seventh and eighth triangular numbers?
18. What is the positive difference between the 19th and 20th triangular numbers?
19. What is the product of the 8th and 9th triangular numbers?
20. What is the product of the 5th and 6th triangular numbers?

This document was prepared by Doug Ray for PSIA, 2002-2003. If you have any questions about the material presented, please email doug@academicmeet.com.